Problem 2.5.2. In a mathematics contest with three problems, 80% of the participants solved the first problem, 75% solved the second and 70% solved the third. Prove that at least 25% of the participants solved *all* three problems.

Problem 2.5.3. Let $\varphi(n)$ denote Euler's totient function, the number of positive integers less than or equal to n which are relatively prime with n. If $p_1, ..., p_k$ are the distinct prime factors of n, prove that

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_k}\right).$$

Problem 2.5.4. Define the Möbius function $\mu : \mathbb{N} :\to \mathbb{R}$ by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is squarefree with } k \geq 0 \text{ distinct prime factors,} \\ 0 & \text{if } n \text{ is not squarefree.} \end{cases}$$

Prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Problem 2.5.5. Given a function $f : \mathbb{N} \to \mathbb{R}$, let $F : \mathbb{N} \to \mathbb{R}$ be

$$F(n) = \sum_{d|n} f(d).$$

Prove that

$$f(n) = \sum_{d|n} F(d)\mu\left(\frac{n}{d}\right).$$

This is called the **Möbius inversion formula**. It is a very important tool in number theory.

Problem 2.5.6. Prove that the number $s_{m,n}$ of surjective functions $f : X \to Y$, where $|X| = n$ and $|Y| = m$ is given by the expression

$$s_{m,n} = m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \ldots + (-1)^{m-1}\binom{m}{m-1}1^n.$$

Problem 2.5.7. Consider 3 sets X, Y, Z with $|X| = n$, $|Y| = m$, $|Z| = r$ and $Z \subset Y$. Denote by $s_{m,n,r}$ the number of functions $f : X \to Y$ for which $Z \subseteq f(X)$. Prove that:

$$s_{m,n,r} = m^n - \binom{r}{1}(m-1)^n + \binom{r}{2}(m-2)^n - \ldots + (-1)^r(m-r)^n.$$

Problem 2.5.8. How many ways are there to seat n married couples at a round table with $2n$ chairs in such a way that the couples never sit next to each other?

Problem 2.5.9. [rook polynomials] Let B be a subset of $[n] \times [n]$. If $j \in \{0, 1, ..., n\}$ we let N_j be the number of permutations σ of $[n]$ such that exactly j elements of the set $\{(i, \sigma(i)) | i \in [n]\}$ belong to B. Also, we let r_k be the number of ways to place k non attacking rooks on B, i.e. the number of k-subsets of B such that no two elements have a common coordinate.

a) Prove that

$$\sum_{j=k}^{n} \binom{j}{k} N_j = r_k (n-k)!$$

for all k.

b) Deduce that

$$\sum_{j=0}^{n} N_j X^j = \sum_{k=0}^{n} r_k (n-k)!(X-1)^k$$

and so

$$N_0 = \sum_{k=0}^{n} (-1)^k r_k (n-k)!.$$

Problem 2.5.10. Show that the number of ways of seating n couples around a table, so that no one sits next to his or her partner is

$$m_n = 2n \cdot \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot 2^k \cdot (2n-k-1)!.$$

Problem 2.5.11. Arguing as in the previous problem, give a new solution to the Crazy Dog Owners Problem.

Problem 2.5.12. Let $A_1, A_2, \ldots, A_n$ $(n \geq 3)$ be finite sets of positive integers. Prove, that

$$\frac{1}{n}\left(\sum_{i=1}^{n} |A_i|\right) + \frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \geq \frac{2}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

holds, where $|E|$ is the cardinality of the set E.

Problem 2.5.13. Let A_1, A_2, $\ldots$, A_n be finite subsets of some set S. Let $d(n)$ be the number of elements which appear exactly in an odd number of sets among A_1, A_2, $\ldots$, A_n. Prove that for any k, $1 \leq k \leq n$ the number

$$d(n) - \sum_{i=1}^{n} |A_i| + 2\sum_{i<j} |A_i \cap A_j| + ... + (-1)^k 2^{k-1} \sum_{i_1 < ... < i_k} |A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}|$$

is divisible by 2^k.

Exercise 2.5.14. Let $a_1, a_2, ..., a_k$ be all positive integers less than n and relatively prime to n. Compute $a_1^2 + a_2^2 + ... + a_k^2$ in terms of n and its prime factorization.

Problem 2.5.15. Let $|U|$, $\sigma(U)$ and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set U of positive integers. (If U is the empty set, $|U| = 0$, $\sigma(U) = 0$, $\pi(U) = 1$.) Let S be a finite set of positive integers. As usual, let $\binom{n}{k}$ denote $\frac{n!}{k!(n-k)!}$. Prove that

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$

for all integers $m \geq \sigma(S)$.

Problem 2.5.16. 25 little donkeys stand in a row; the rightmost of them is Eeyore. Winnie-the-Pooh wants to give a balloon of one of the seven colours of the rainbow to each donkey, so that successive donkeys receive balloons of different colors, and so that at least one balloon of each color is given to some donkey. Eeyore wants to give to each of the 24 remaining donkeys a pot of one of six colors of the rainbow (except red), so that at least one pot of each color is given to some donkey (but successive donkeys can receive pots of the same color). Whom of the two friends has more ways to get his plan implemented, and how many times more?